

Schrödinger quantization of linearly polarized Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models coupled to massless scalar fields

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In this paper we will construct the Schrödinger representation for the linearly polarized Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models coupled to massless scalar fields. Here the quantum states belong to a L^2 -space for a suitable quantum configuration space endowed with a Gaussian measure, whose support is analyzed. This study completes the quantization of these systems previously performed in the Fock scheme, and provides a specially useful framework to address physically relevant questions.

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I. INTRODUCTION

Gowdy models are $U(1) \times U(1)$ symmetry reductions with many interesting applications in cosmology and quantum gravity, since they provide inhomogeneous systems with local degrees of freedom and invariance under a certain class of diffeomorphisms [1].

The exact quantization of the linearly polarized Gowdy \mathbb{T}^3 model in the vacuum has been profusely analyzed [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Its gravitational local degrees of freedom can be interpreted as those corresponding to a massless scalar field in a fiducial background, so that the usual techniques of QFT in curved spacetimes can be applied in order to construct the quantum theory. The fact that the linear symplectic transformations describing the classical time evolution cannot be unitarily implemented in the physical Hilbert space when the system is written in terms of its original variables was initially interpreted as a serious obstacle for the feasibility of the model [6]. Nevertheless, it is possible to overcome this problem by a suitable time-dependent redefinition of the field [9]. Furthermore, by demanding the unitarity of the dynamics and the invariance under an extra $U(1)$ symmetry generated by a residual global constraint, the existence of a unique (up to unitary equivalence) Fock representation can be proved for the system [11, 12].

The existing literature has been recently extended to the remaining topologies, $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 , allowing the coupling of gravity to massless scalar fields (see [15] for a rigorous classical treatment of these models). Here, both gravitational and matter local degrees of freedom can be encoded by massless scalar fields evolving in the same fixed background metric. Therefore, they can be treated in a unified way for the construction of the quantum theory. A re-scaling of the fields similar to the one defined in the three-torus case permits also a unitary implementation of the dynamics [16]. Concretely, this redefinition is dictated

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by the conformal factor $\sin t$ that relates the Gowdy metrics to the Einstein static (1+2)-universe. For these models, at variance with the three-torus case, there is no extra constraint, so that one obtains a family of (in general) unitarily nonequivalent Fock representations, and in principle there is no symmetry argument to select a preferred one. However, the uniqueness of the representation can be recovered in these cases by imposing the unitarity of the dynamics and the $SO(3)$ invariance of the Fock construction¹ [16]. Furthermore, it is expected that a discussion similar to the one developed in [12] for the vacuum Gowdy \mathbb{T}^3 model will lead us to conclude that this redefinition of the fields is the only reasonable one (up to multiplicative constants) providing unitary dynamics under the condition of $SO(3)$ invariance.

We will consider the Schrödinger representation for the linearly polarized Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models coupled to massless scalar fields, where the states act as functionals on the quantum configuration space \mathcal{C} for a fixed time t_0 . Here $\overline{\mathcal{C}}$ is an appropriate distributional extension of the classical configuration space \mathcal{C} , taken in these cases to be the space of tempered distributions on the 2-sphere. The Hilbert space then takes the form $\mathcal{H}_s(t_0) = L^2(\overline{\mathcal{C}}, d\mu_{t_0})$. The identification of the Gaussian nature of the measure μ_{t_0} , the non-standard representation of the momentum operator, and the relation between Schrödinger and Fock representations were exhaustively analyzed in [18] as a natural extension to the functional description of the Fock quantization of scalar fields in curved backgrounds [19]. In the QFT context, the Schrödinger representation has been historically pushed into the background in favor of the usual Fock one because of the difficulty in using it to address sensible questions regarding physical scattering processes. However, it is certainly the most natural representation in the context of canonical quantum gravity, in view of the splitting of spacetime into spatial sections of constant time. Furthermore, as was pointed out in [13] for the vacuum three-torus case, it provides a better understanding of the properties of the quantized field, since it is possible to determine the behavior of the typical field configurations through the study of the measure support. The Schrödinger representation is also a privileged framework to probe the existence and properties of semiclassical and squeezed states for these systems. This paper represents then a necessary first step to tackle this issue that will be considered elsewhere after the rigorous analysis of this type of quantizations [20]. Note that this is not a trivial question owing to the nonautonomous nature of the Hamiltonian that governs the reduced dynamics of the models.

In this paper, we will closely rely on the notation and results of [15, 16], where the reader can find the classical and quantum formulations of the Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models, as well as on the recent works [13, 14], devoted to the Schrödinger representation for the vacuum Gowdy \mathbb{T}^3 model both for the original and the redefined scalar fields, respectively. In particular, the need for extending the results found for the three-torus case to the remaining topologies, and discussing the differences between them, was already pointed out in [13]. In section II we will summarize the features of the Fock construction for the Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models corresponding to the re-scaled fields for which the dynamics is unitary, analyzing in subsection IIB the implementation of the Hamiltonian as a self-adjoint operator for

¹ See also [17] for an independent proof of this result. In this reference, some problems concerning the completeness of the results given in [16] were pointed out. Nevertheless, they can be easily solved by introducing some minor changes that will be taken into account in the next section.

each value of the time parameter. We will also discuss here the possibility of modifying the expression of the Hamiltonian at the classical level in order to avoid some problems regarding the domain of its quantum counterpart. In section III, we will proceed to define the Schrödinger representation for these models in such a way that the construction is unitarily equivalent to the Fock one. In particular, we will probe the properties and support of the measure μ_{t_0} in subsection III B, as well as the representation of the canonical commutation relations in subsection III C. In section IV, we will check that, as a consequence of the unitary implementation of the time evolution, the representations corresponding to different values of the time parameter are unitarily equivalent, and also that their associated measures are mutually absolutely continuous. Finally, in section V, we will make some comments and remarks on the results of the paper, in particular concerning their similarity with those found for the three-torus case.

II. FOCK REPRESENTATION

A. General framework

The dynamics of both gravitational and matter local degrees of freedom² in the linearly polarized Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models can be described by the same nonautonomous Hamiltonian system $(\mathbf{P}, \omega, H(t))$, whose features we proceed to summarize. Let $\gamma_{ab} = (d\theta)_a(d\theta)_b + \sin^2 \theta (d\sigma)_a(d\sigma)_b$ be the round metric in the 2-sphere \mathbb{S}^2 , with spherical coordinates $(\theta, \sigma) \in (0, \pi) \times (0, 2\pi)$. \mathbf{P} is the space of smooth and symmetric Cauchy data $(Q, P) \in C^\infty(\mathbb{S}^2; \mathbb{R}) \times C^\infty(\mathbb{S}^2; \mathbb{R})$, with $\mathcal{L}_\sigma Q = 0 = \mathcal{L}_\sigma P$, where \mathcal{L}_σ denotes the Lie derivative with respect to the vector field $\sigma^a = (\partial/\partial\sigma)^a$. The standard (weakly) symplectic structure $\omega : \mathbf{P} \times \mathbf{P} \rightarrow \mathbb{R}$ is given by

$$\omega((Q_1, P_1), (Q_2, P_2)) := \int_{\mathbb{S}^2} |\gamma|^{1/2} (Q_2 P_1 - Q_1 P_2), \quad (Q_1, P_1), (Q_2, P_2) \in \mathbf{P}. \quad (2.1)$$

The symplectic space (\mathbf{P}, ω) is then the *canonical phase space* of the system. Finally, $H : (0, \pi) \times \mathbf{P} \rightarrow \mathbb{R}$ is the (indefinite) nonautonomous Hamiltonian

$$H(t; Q, P) := \frac{1}{2} \int_{\mathbb{S}^2} |\gamma|^{1/2} (P^2 + \cot t QP - Q \Delta_{\mathbb{S}^2} Q), \quad (2.2)$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on the round 2-sphere. Consider now the space of smooth and symmetric real solutions to the Euler-Lagrange equation derived from the Hamilton equations³

$$\mathcal{S} := \left\{ \xi \in C^\infty((0, \pi) \times \mathbb{S}^2; \mathbb{R}) \mid -\ddot{\xi} + \Delta_{\mathbb{S}^2} \xi = \frac{1}{4} (1 + \csc^2 t) \xi, \quad \mathcal{L}_\sigma \xi = 0 \right\}. \quad (2.3)$$

² In what follows we will consider the use of the redefined scalar field ξ for which the dynamics can be unitarily implemented [16]. We will not study the global modes present in these models [15]. They can be quantized in a straightforward way in terms of standard position and momentum operators with dense domain in $L^2(\mathbb{R})$.

³ The dot denotes time derivative.

We define the *covariant phase space* of the system as the pair (\mathcal{S}, Ω) , where $\Omega : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ is the symplectic structure naturally induced by the ω given in (2.1),

$$\Omega(\xi_1, \xi_2) := \int_{\mathbb{S}^2} |\gamma|^{1/2} \iota_t^* (\xi_2 \dot{\xi}_1 - \xi_1 \dot{\xi}_2), \quad \xi_1, \xi_2 \in \mathcal{S}, \quad (2.4)$$

with $\iota_t : \mathbb{S}^2 \rightarrow (0, \pi) \times \mathbb{S}^2$ being the embedding of the 2-sphere as a Cauchy surface of constant time t .

In order to obtain the quantum theory for these models, it is necessary to construct the one-particle Hilbert space of the system $\mathcal{H}_{\mathcal{P}}$. Consider the Lagrangian subspace

$$\mathcal{P} := \left\{ Z \in \mathcal{S}_{\mathbb{C}} \mid Z = \sum_{\ell=0}^{\infty} a_{\ell} z_{\ell} Y_{\ell 0}, \quad a_{\ell} \in \mathbb{C} \right\} \quad (2.5)$$

of the complexification $\mathcal{S}_{\mathbb{C}}$ of the solution space \mathcal{S} , where $(z_{\ell})_{\ell=0}^{\infty}$ is a family of complex linearly independent solutions to the equation

$$\ddot{z}_{\ell} + \left(\frac{1}{4} (1 + \csc^2 t) + \ell(\ell + 1) \right) z_{\ell} = 0 \quad (2.6)$$

satisfying the normalization condition⁴

$$z_{\ell} \dot{\bar{z}}_{\ell} - \bar{z}_{\ell} \dot{z}_{\ell} = i, \quad (2.7)$$

and $Y_{\ell 0}$ are the spherical harmonics verifying the orthogonality conditions $\int_{\mathbb{S}^2} |\gamma|^{1/2} Y_{\ell 0} Y_{\ell' 0} = \delta(\ell, \ell')$. The one-particle Hilbert space $\mathcal{H}_{\mathcal{P}}$ is then the Cauchy completion of the subspace \mathcal{P} with respect to the inner product

$$\langle Z_1 \mid Z_2 \rangle_{\mathcal{P}} := -i \Omega_{\mathbb{C}}(\bar{Z}_1, Z_2) = \sum_{\ell=0}^{\infty} \bar{a}_{\ell}^{(1)} a_{\ell}^{(2)}, \quad Z_1, Z_2 \in \mathcal{P}, \quad (2.8)$$

where $\Omega_{\mathbb{C}}$ is the extension of the symplectic structure (2.4) to $\mathcal{S}_{\mathbb{C}}$ by linearity. Finally, the Hilbert space⁵ of the models is given by the symmetric Fock space defined on $\mathcal{H}_{\mathcal{P}}$,

$$\mathcal{F}_{\mathcal{P}} := \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathcal{P}}^{\otimes n}, \quad (2.9)$$

where $\mathcal{H}_{\mathcal{P}}^{\otimes n}$ denotes the subspace of $\mathcal{H}_{\mathcal{P}}^{\otimes n} = \otimes_{k=1}^n \mathcal{H}_{\mathcal{P}}$ spanned by symmetric tensor products of n vectors in $\mathcal{H}_{\mathcal{P}}$. The possible choices of Lagrangian subspaces \mathcal{P} are encoded in the following two-parameter family of z_{ℓ} functions satisfying (2.7):

$$z_{\ell}(t) = \sqrt{\frac{\sin t}{2}} \left(\rho_{\ell} \mathcal{P}_{\ell}(\cos t) + (\nu_{\ell} + i \rho_{\ell}^{-1}) \mathcal{Q}_{\ell}(\cos t) \right), \quad (2.10)$$

⁴ The bar denotes complex conjugation.

⁵ The unnecessary distinction between kinematical and physical Hilbert spaces in these models follows from the nonexistence of extra constraints [15].

with $\rho_\ell > 0$, $\nu_\ell \in \mathbb{R}$, modulo a multiplicative phase that plays no role in the context of the study of unitary implementation of dynamics. \mathcal{P}_ℓ and \mathcal{Q}_ℓ denote the first and second class Legendre functions, respectively.

Every election of \mathcal{P} is in one-to-one correspondence with a Ω -compatible $SO(3)$ -invariant complex structure on \mathcal{S} , $J_{\mathcal{P}} : \mathcal{S} \rightarrow \mathcal{S}$, $J_{\mathcal{P}}^2 = -\text{Id}_{\mathcal{S}}$ (see [16]). Indeed, any solution $\xi \in \mathcal{S}$ can be uniquely decomposed as $\xi = Z + \bar{Z}$, with $Z \in \mathcal{P}$, in such a way that $J_{\mathcal{P}}$ is defined as

$$J_{\mathcal{P}}\xi := i(Z - \bar{Z}). \quad (2.11)$$

As proved in [16], the linear symplectic transformations that describe the time evolution can be unitarily implemented in the Hilbert space $\mathcal{F}_{\mathcal{P}}$ for all those $SO(3)$ -invariant complex structures $J_{\mathcal{P}}$ characterized by pairs $(\rho_\ell, \nu_\ell)_{\ell=0}^\infty$ such that⁶

$$\rho_\ell = \sqrt{\frac{\pi}{2}} + x_\ell > 0, \quad (x_\ell)_{\ell=0}^\infty \in \ell^2(\mathbb{R}), \quad \text{and} \quad (\nu_\ell)_{\ell=0}^\infty \in \ell^2(\mathbb{R}). \quad (2.12)$$

In addition, all the Fock representations obtained through (2.12) are unitarily equivalent [16, 17]. In the following, we will implicitly assume the use of a concrete complex structure $J_{\mathcal{P}}$ of this type.

B. Self-adjointness of the quantum Hamiltonian

Note that due to the nonautonomous nature of the classical Hamiltonian (2.2), the dynamics does not define a one-parameter symplectic group on (\mathbf{P}, ω) , so we cannot apply Stone's theorem to justify the self-adjointness of the corresponding (one-parameter family of) operators in the quantum theory. Nevertheless, it is possible to show that the quantum Hamiltonian is self-adjoint for each value of the time parameter t by analyzing the unitary implementability on $\mathcal{F}_{\mathcal{P}}$ of the one-parameter symplectic group generated by the *autonomous* Hamiltonian $H(\tau)$, once a value $t = \tau \in (0, \pi)$ has been fixed. Here, we will follow the efficient procedure employed in [7] for the Gowdy \mathbb{T}^3 model, subsequently generalized in [10] to discuss the self-adjointness of general quadratic operators in this context. We start by considering the auxiliary system $(\mathbf{P}, \omega, H(\tau))$, where the dynamics is governed by the classical autonomous Hamiltonian

$$H(\tau) = \frac{1}{2} \sum_{\ell=0}^{\infty} (K_\ell(\tau)a_\ell^2 + \bar{K}_\ell(\tau)\bar{a}_\ell^2 + 2G_\ell(\tau)\bar{a}_\ell a_\ell), \quad (2.13)$$

⁶ The expression of the ρ_ℓ coefficients appearing in [16] is incomplete, and needs to be corrected by taking into account the subdominant term that appears in (2.12). With more generality, the expression of ν_ℓ given in [16] must also be replaced by the one of equation (2.12) in order to explicitly include nonpolynomial decreasing behaviors. Taking these minor changes into consideration, we completely characterize the biparametric family of complex structures for which dynamics is unitary, and not only a subfamily as in [16], solving the problems pointed out at the end of reference [17]. We must remark, in any case, that these corrections do not affect the main conclusions achieved in [16], in particular, concerning the proof of uniqueness of the Fock representation, whose simplicity typifies the usefulness of the formalism developed in [16].

with

$$\begin{aligned} K_\ell(\tau) &:= \left(\dot{z}_\ell(\tau) - \frac{1}{2} \cot \tau z_\ell(\tau) \right)^2 + \ell(\ell+1) z_\ell^2(\tau) + \cot \tau \left(\dot{z}_\ell(\tau) - \frac{1}{2} \cot \tau z_\ell(\tau) \right) z_\ell(\tau), \\ G_\ell(\tau) &:= \left| \dot{z}_\ell(\tau) - \frac{1}{2} \cot \tau z_\ell(\tau) \right|^2 + \ell(\ell+1) |z_\ell(\tau)|^2 \\ &\quad + \frac{1}{2} \cot \tau \left(\left(\dot{z}_\ell(\tau) - \frac{1}{2} \cot \tau z_\ell(\tau) \right) \bar{z}_\ell(\tau) + \left(\dot{\bar{z}}_\ell(\tau) - \frac{1}{2} \cot \tau \bar{z}_\ell(\tau) \right) z_\ell(\tau) \right). \end{aligned} \quad (2.14)$$

The modes a_ℓ, \bar{a}_ℓ are defined through the relations $Q_\ell := \int_{\mathbb{S}^2} |\gamma|^{1/2} Q Y_{\ell 0} = z_\ell(\tau) a_\ell + \bar{z}_\ell(\tau) \bar{a}_\ell$, $P_\ell := \int_{\mathbb{S}^2} |\gamma|^{1/2} P Y_{\ell 0} = \left(\dot{z}_\ell(\tau) - (1/2) \cot \tau z_\ell(\tau) \right) a_\ell + \left(\dot{\bar{z}}_\ell(\tau) - (1/2) \cot \tau \bar{z}_\ell(\tau) \right) \bar{a}_\ell$. Their evolution in a fictitious time parameter $s \in \mathbb{R}$ is given by the linear equations⁷

$$\begin{aligned} \frac{da_\ell}{ds} &= \{a_\ell, H(\tau)\} = -i(G_\ell(\tau) a_\ell + \bar{K}_\ell(\tau) \bar{a}_\ell), \\ \frac{d\bar{a}_\ell}{ds} &= \{\bar{a}_\ell, H(\tau)\} = i(K_\ell(\tau) a_\ell + G_\ell(\tau) \bar{a}_\ell). \end{aligned} \quad (2.15)$$

Using the normalization condition (2.7), we easily obtain the second-order differential equation

$$\frac{d^2 a_\ell}{ds^2} = -\left(\ell(\ell+1) - \frac{1}{4} \cot^2 \tau \right) a_\ell, \quad (2.16)$$

whose solutions have a linear dependence on the initial conditions $a_\ell(s_0)$ and $\bar{a}_\ell(s_0)$,

$$a_\ell(s) = \alpha_\ell(s, s_0) a_\ell(s_0) + \beta_\ell(s, s_0) \bar{a}_\ell(s_0), \quad \bar{a}_\ell(s) = \overline{\alpha_\ell(s, s_0)} \bar{a}_\ell(s_0) + \overline{\beta_\ell(s, s_0)} a_\ell(s_0). \quad (2.17)$$

This symplectic transformation is unitarily implementable on $\mathcal{F}_\mathcal{P}$ for each $s \in \mathbb{R}$, i.e., there exists a unitary operator $\hat{u}(s, s_0) : \mathcal{F}_\mathcal{P} \rightarrow \mathcal{F}_\mathcal{P}$ such that $\hat{u}(s, s_0) \hat{a}_\ell \hat{u}^{-1}(s, s_0) = \alpha_\ell(s, s_0) \hat{a}_\ell + \beta_\ell(s, s_0) \hat{a}_\ell^\dagger$, $\hat{u}(s, s_0) \hat{a}_\ell^\dagger \hat{u}^{-1}(s, s_0) = \bar{\beta}_\ell(s, s_0) \hat{a}_\ell + \bar{\alpha}_\ell(s, s_0) \hat{a}_\ell^\dagger$, if and only if the Bogoliubov coefficients β_ℓ are square summable [21],

$$\sum_{\ell=0}^{\infty} |\beta_\ell(s, s_0)|^2 < +\infty. \quad (2.18)$$

Note that, for each value of $\tau \in (0, \pi)$, there exists $\ell_0 \in \mathbb{N} \cup \{0\}$ such that

$$\lambda_\ell^2 := \ell(\ell+1) - \frac{1}{4} \cot^2 \tau > 0, \quad \forall \ell > \ell_0.$$

In this situation,

$$\begin{aligned} \alpha_\ell(s, s_0) &= \cos(\lambda_\ell(s - s_0)) - i \lambda_\ell^{-1} G_\ell(\tau) \sin(\lambda_\ell(s - s_0)), \\ \beta_\ell(s, s_0) &= -i \lambda_\ell^{-1} \bar{K}_\ell(\tau) \sin(\lambda_\ell(s - s_0)). \end{aligned}$$

It suffices to consider the modes corresponding to $\ell > \ell_0$, since the convergence of the series (2.18) depends, in practice, only on the high-frequency behavior of the β_ℓ coefficients. Taking

⁷ Here $\{\cdot, \cdot\}$ denotes the Poisson bracket defined from (2.1), with $\{a_\ell, \bar{a}_{\ell'}\} = -i\delta(\ell, \ell')\mathbb{I}$.

into account the asymptotic expansions in ℓ

$$\begin{aligned} z_\ell(t) &= \frac{1}{\sqrt{2\ell}} \exp(-i[(\ell + 1/2)t - \pi/4]) + O(\ell^{-3/2}), \\ \dot{z}_\ell(t) - \frac{1}{2} \cot t z_\ell(t) &= -i\sqrt{\frac{\ell}{2}} \exp(-i[(\ell + 1/2)t - \pi/4]) + O(\ell^{-1/2}), \end{aligned} \quad (2.19)$$

we have $K_\ell(\tau) = O(1)$, so that $\sum_{\ell > \ell_0} \lambda_\ell^{-2} |K_\ell(\tau)|^2 \sin^2(\lambda_\ell(s - s_0)) < +\infty$, $\forall s \in \mathbb{R}$, and hence (2.18) is verified. Finally, the transformation (2.17) is implementable as a continuous, unitary, one-parameter group if it verifies the strong continuity condition in the auxiliary parameter s

$$\lim_{s \rightarrow s_0} \sum_{\ell=0}^{\infty} |a_\ell(s) - a_\ell(s_0)|^2 = 0, \quad s_0 \in \mathbb{R}. \quad (2.20)$$

Again, we can restrict ourselves to the modes $\ell > \ell_0$. It is straightforward to check that this condition holds for the solution (2.17) with square summable initial data a_ℓ and \bar{a}_ℓ . Therefore, we have obtained a strongly continuous and unitary one-parameter group whose generator is self-adjoint according to Stone's theorem.

The quantum Hamiltonian of the models under consideration can be explicitly calculated as the strong limit

$$\text{s-lim}_{t' \rightarrow t} \frac{\hat{U}(t, t') - \hat{\mathbb{I}}}{t - t'} f = -i\hat{H}(t)f, \quad f \in \mathcal{D}_{\hat{H}(t)},$$

where $\hat{U}(t, t')$ denotes the quantum evolution operator on $\mathcal{F}_{\mathcal{P}}$. The previous result ensures the self-adjointness of the quantum Hamiltonian $\hat{H}(t)$, and hence the existence of a dense domain $\mathcal{D}_{\hat{H}(t)} \subset \mathcal{F}_{\mathcal{P}}$, for each value of the time parameter $t \in (0, \pi)$. Unfortunately, the method employed does not provide us with a characterization of such domains, or the concrete expression of the quantum Hamiltonian. Nevertheless, given the quadratic nature of the classical Hamiltonian (2.2), it is expected that this limit coincides with the operator directly promoted from the classical function up to normal ordering. As proved in [16], this operator does not have the Fock vacuum state $|0\rangle_{\mathcal{P}} := 1 \oplus 0 \oplus 0 \oplus \dots \in \mathcal{F}_{\mathcal{P}}$ in its domain because of the fact that the K_ℓ sequence defined in (2.14) is not square summable. As a consequence, the action of the operator is not defined either on the dense subspace of states with a finite number of particles. This difficulty can be overcome right from the start by describing the classical dynamics through the (positive definite) Hamiltonian [16]

$$H_0(Q, P; t) := \frac{1}{2} \int_{\mathbb{S}^2} |\gamma|^{1/2} \left(P^2 + Q \left[\frac{1}{4} (1 + \csc^2 t) - \Delta_{\mathbb{S}^2} \right] Q \right). \quad (2.21)$$

The Hamiltonians (2.2) and (2.21) obviously govern the same classical evolution, but they are connected by a time-dependent symplectic transformation that in principle is not unitarily implementable, so one possibly obtains nonequivalent quantum theories from them. The corresponding quantum Hamiltonian is given, after normal ordering, by

$$\hat{H}_0(t) = \frac{1}{2} \sum_{\ell=0}^{\infty} \left(K_{0_\ell}(t) \hat{a}_\ell^2 + \bar{K}_{0_\ell}(t) \hat{a}_\ell^{\dagger 2} + 2G_{0_\ell}(t) \hat{a}_\ell^\dagger \hat{a}_\ell \right), \quad (2.22)$$

where

$$\begin{aligned} K_{0_\ell}(t) &:= \dot{z}_\ell^2(t) + \left(\frac{1}{4}(1 + \csc^2 t) + \ell(\ell + 1) \right) z_\ell^2(t), \\ G_{0_\ell}(t) &:= |\dot{z}_\ell(t)|^2 + \left(\frac{1}{4}(1 + \csc^2 t) + \ell(\ell + 1) \right) |z_\ell(t)|^2. \end{aligned} \quad (2.23)$$

Here, \hat{a}_ℓ^\dagger and \hat{a}_ℓ are the creation and annihilation operators associated with the modes $z_\ell Y_{\ell 0}$, respectively. This new self-adjoint Hamiltonian has the advantage of including the vacuum state in its domain –in this case $K_{0_\ell}(t)$ defines a square summable sequence for each value of t –, as well as the fact that the results about the unitary implementation of the time evolution and the uniqueness of the Fock representation are also valid in this case. Concretely, the biparametric family of complex structures for which the dynamics is unitary is characterized again by the pairs (2.12). In what follows, we will consider the dynamics of the system to be described by (2.21).

III. SCHRÖDINGER REPRESENTATION

A. Constructing the L^2 space

Let us denote by \mathcal{S} the Schwartz space of smooth and symmetric test functions on the 2-sphere,

$$\mathcal{S} := \{f \in C^\infty(\mathbb{S}^2; \mathbb{R}) \mid \mathcal{L}_\sigma f = 0\}, \quad (3.1)$$

endowed with the standard nuclear topology⁸. The quantum configuration space used to define the Schrödinger representation is then the topological dual \mathcal{S}' , consisting of continuous linear functionals on \mathcal{S} . Note that this space includes the delta functions and their derivatives. Given a time of embedding t_0 , the Schrödinger representation is introduced by defining a suitable Hilbert space⁹ $L^2(\mathcal{S}', d\mu_{t_0})$, for a certain measure μ_{t_0} , in which the configuration observables act as *multiplication* operators. As we will see later, given the Gaussian nature of the measure μ_{t_0} , the momentum operators will differ from the usual ones in terms of derivatives by a multiplicative term depending on the configuration variables.

As a consequence of the linearity of $\mathbf{P} = \mathcal{S} \times \mathcal{S}$, the set of elementary classical observables \mathcal{O} can be identified with the \mathbb{R} -vector space generated by linear functionals on \mathbf{P} . Every pair $\lambda := (-g, f) \in \mathbf{P}$, $f, g \in \mathcal{S}$, has an associated functional $F_\lambda : \mathbf{P} \rightarrow \mathbb{R}$ such that for all $X = (Q, P) \in \mathbf{P}$,

$$F_\lambda(X) := \omega(\lambda, X) = \int_{\mathbb{S}^2} |\gamma|^{1/2} (fQ + gP). \quad (3.2)$$

⁸ Every element $f \in \mathcal{S}$ can be expanded as $f(s) = \sum_{\ell=0}^{\infty} f_\ell Y_{\ell 0}(s)$, $s \in \mathbb{S}^2$, with $(f_\ell)_{\ell=0}^{\infty}$ a sequence of rapidly decreasing real coefficients, such that $\lim_{\ell \rightarrow \infty} \ell^n f_\ell = 0$, $\forall n \in \mathbb{N} \cup \{0\}$. We will revise the equivalent description of the topological structure of \mathcal{S} in terms of the locally convex space of rapidly decreasing sequences in section IIIB. For more details, the reader can consult [22].

⁹ Here, the measure μ_{t_0} is implicitly assumed to be defined on the sigma algebra $\sigma(\text{Cyl}(\mathcal{S}'))$ generated by the cylinder sets.

Therefore, $\mathcal{O} = \text{Span}\{\mathbb{I}, F_\lambda\}_{\lambda \in \mathbf{P}}$. As expected [23], this set satisfies the condition that any regular function on \mathbf{P} can be obtained as a (suitable limit of) sum of products of elements in \mathcal{O} , and also that it is closed under Poisson brackets, $\{F_\lambda(\cdot), F_\nu(\cdot)\} = F_\nu(\lambda)\mathbb{I}$. The configuration and momentum observables are objects of this type defined by the pairs $\lambda = (0, f)$ and $\lambda = (-g, 0)$, respectively

$$Q(f) := F_{(0,f)}(Q, P) = \int_{\mathbb{S}^2} |\gamma|^{1/2} f Q = \sum_{\ell=0}^{\infty} f_\ell Q_\ell, \quad (3.3)$$

$$P(g) := F_{(-g,0)}(Q, P) = \int_{\mathbb{S}^2} |\gamma|^{1/2} g P = \sum_{\ell=0}^{\infty} g_\ell P_\ell, \quad (3.4)$$

where the symmetric test functions have been expanded as explained in footnote 8. Here, with the aim of simplifying the notation, we have used the same symbol to denote the canonical inclusion $\mathcal{S} \hookrightarrow \mathcal{S}'$ of \mathcal{S} into \mathcal{S}' . In this way, $F_{(-g,f)}(Q, P) = Q(f) + P(g)$. The abstract quantum algebra of observables \mathcal{A} is then given by the usual Weyl C^* -algebra generated by the elements $W(\lambda) = \exp(iF_\lambda)$, $\lambda \in \mathbf{P}$, satisfying the conditions

$$W(\lambda)^* = W(-\lambda), \quad W(\lambda_1)W(\lambda_2) = e^{\frac{i}{2}\omega(\lambda_1, \lambda_2)} W(\lambda_1 + \lambda_2), \quad (3.5)$$

containing the information about the canonical commutation relations.

Let $\mathfrak{J}_{t_0} : \mathbf{P} \rightarrow \mathcal{S}$, $t_0 \in (0, \pi)$, be the symplectomorphism that defines for each pair of Cauchy data $(Q, P) \in \mathbf{P}$ the unique solution $\xi \in \mathcal{S}$ such that, under the evolution given by the Hamiltonian (2.21), it satisfies $\xi(t_0, s) = Q(s)$, $\dot{\xi}(t_0, s) = P(s)$. That is

$$\xi(t, s) = (\mathfrak{J}_{t_0}(Q, P))(t, s) = \sum_{\ell=0}^{\infty} \left(a_\ell(t_0) z_\ell(t) + \overline{a_\ell(t_0) z_\ell(t)} \right) Y_{\ell 0}(s) \in \mathcal{S}, \quad (3.6)$$

with

$$a_\ell(t_0) := i\bar{z}_\ell(t_0)P_\ell - i\dot{\bar{z}}_\ell(t_0)Q_\ell. \quad (3.7)$$

This map gives rise to a natural ω -compatible complex structure on the canonical phase space given by

$$J_{t_0} := \mathfrak{J}_{t_0}^{-1} \circ J_{\mathcal{P}} \circ \mathfrak{J}_{t_0} : \mathbf{P} \rightarrow \mathbf{P}, \quad (3.8)$$

such that

$$(Q, P) \in \mathbf{P} \mapsto J_{t_0}(Q, P) = (A(t_0)Q + B(t_0)P, D(t_0)Q + C(t_0)P) \in \mathbf{P},$$

where $A(t_0), B(t_0), C(t_0), D(t_0) : \mathcal{S} \rightarrow \mathcal{S}$ are linear operators satisfying, in virtue of the ω -compatibility [24], the relations

$$\langle f, B(t_0)f' \rangle = \langle B(t_0)f, f' \rangle, \quad \langle g, D(t_0)g' \rangle = \langle D(t_0)g, g' \rangle, \quad \langle f, A(t_0)g \rangle = -\langle C(t_0)f, g \rangle,$$

for all $f, g, f', g' \in \mathcal{S}$. Here, we have denoted $\langle f, g \rangle := \int_{\mathbb{S}^2} |\gamma|^{1/2} f g$. Also, given the condition $J_{t_0}^2 = -\text{Id}_{\mathbf{P}}$, and assuming $B(t_0)$ invertible, the $C(t_0)$ and $D(t_0)$ operators can be expressed in terms of the $A(t_0)$ and $B(t_0)$ operators through the relations $C(t_0) = -B^{-1}(t_0)A(t_0)B(t_0)$ and $D(t_0) = -B^{-1}(t_0)(\mathbf{1} + A^2(t_0))$, respectively, in such a way that the complex structure

J_{t_0} is completely characterized by $A(t_0)$ and $B(t_0)$. Using equations (2.11) and (3.6), it is straightforward to obtain¹⁰

$$\begin{aligned} (A(t_0)Q)(s) &= \sum_{\ell=0}^{\infty} (\dot{z}_{\ell}(t_0)z_{\ell}(t_0) + \dot{z}_{\ell}(t_0)\bar{z}_{\ell}(t_0))Q_{\ell}Y_{\ell 0}(s), \\ (B(t_0)P)(s) &= -2 \sum_{\ell=0}^{\infty} |z_{\ell}(t_0)|^2 P_{\ell}Y_{\ell 0}(s). \end{aligned} \quad (3.9)$$

It is worth noting that, given the rapidly decreasing nature of the sequences $(Q_{\ell})_{\ell=0}^{\infty}$ and $(P_{\ell})_{\ell=0}^{\infty}$, as well as the asymptotic behavior of the z_{ℓ} functions decaying like (2.19), the $A(t_0)$ and $B(t_0)$ operators are well defined on \mathcal{S} . In addition, $B(t_0)$ has an inverse operator $B^{-1}(t_0) : \mathcal{S} \rightarrow \mathcal{S}$ given by

$$(B^{-1}(t_0)P)(s) = -\frac{1}{2} \sum_{\ell=0}^{\infty} |z_{\ell}(t_0)|^{-2} P_{\ell}Y_{\ell 0}(s). \quad (3.10)$$

By definition, once a time of embedding t_0 is fixed, the states in the Schrödinger representation are characterized as functionals $\Psi : \mathcal{S}' \rightarrow \mathbb{C}$ belonging to a certain Hilbert space $\mathcal{H}_s(t_0) = L^2(\mathcal{S}', d\mu_{t_0})$. Due to the infinite dimensionality of the quantum configuration space, it is not possible to define a Lebesgue-type measure μ_{t_0} , but rather a probability one¹¹. This representation is constructed in such a way that it is associated with the state $\varpi_{t_0} : \mathcal{A} \rightarrow \mathbb{C}$ on the Weyl algebra \mathcal{A} whose action on the elementary observables is given by [18, 19]

$$\varpi_{t_0}(W(\lambda)) = \exp\left(-\frac{1}{4}\omega(J_{t_0}(\lambda), \lambda)\right), \quad \lambda \in \mathbf{P}. \quad (3.11)$$

We will check in section IV that the Schrödinger representations corresponding to different values of the time parameter are unitarily equivalent due to the unitary implementability of the dynamics. We require that the configuration observables are represented as *multiplication* operators, so that for $\lambda = (0, f) \in \mathbf{P}$,

$$\pi_s(t_0) \cdot W(\lambda)|_{\lambda=(0,f)} = \exp(i\hat{Q}_{t_0}[f]), \quad (\hat{Q}_{t_0}[f]\Psi)[\tilde{Q}] = \tilde{Q}(f)\Psi[\tilde{Q}], \quad (3.12)$$

where $\tilde{Q} \in \mathcal{S}'$ denotes a generic distribution of \mathcal{S}' and $\tilde{Q}(f)$ gives the usual pairing between \mathcal{S} and \mathcal{S}' , $\Psi \in \mathcal{D}_{\hat{Q}_{t_0}[f]} \subset \mathcal{H}_s(t_0)$ (the self-adjointness of the configuration and momentum operators will be discussed in subsection III C), and $\pi_s(t_0) : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_s(t_0))$ is the map from the Weyl algebra \mathcal{A} to the collection of bounded linear operators on $\mathcal{H}_s(t_0)$. In this way, the measure μ_{t_0} is Gaussian with covariance $\mathcal{C}(t_0) := -B(t_0)/2$, and thus its Fourier

¹⁰ Note that the zero mode $\ell = 0$ has been included into the spherical harmonic expansion of the test functions. The $B(t_0)$ operator is well defined even for this mode, ultimately as a consequence of equation (2.6) verified by the z_{ℓ} functions, where the squared frequency is positive definite $\forall t \in (0, \pi)$ when $\ell = 0$.

¹¹ This is, a measure satisfying $\int_{\mathcal{S}'} d\mu_{t_0} = 1$.

transform is given by¹²

$$\int_{\mathcal{S}'} e^{i\tilde{Q}(f)} d\mu_{t_0}[\tilde{Q}] = \exp\left(\frac{1}{4}\langle f, B(t_0)f \rangle\right). \quad (3.13)$$

The covariance operator $\check{\mathcal{C}}_{t_0} : \mathbf{P} \rightarrow \mathbb{R}$ is defined as $\check{\mathcal{C}}_{t_0}(f, g) := \langle f, \mathcal{C}(t_0)g \rangle$, $f, g \in \mathcal{S}$. Since $|z_\ell(t_0)|^2$ is bounded and positive definite $\forall t \in (0, \pi)$ and $\forall \ell \in \mathbb{N} \cup \{0\}$, it follows that, as expected, $\check{\mathcal{C}}_{t_0}$ is a nondegenerate positive definite and continuous bilinear form on the topological vector space \mathcal{S} .

B. Properties of the measure

In order to easily visualize the nature of the measure μ_{t_0} , note that upon restriction on any number of coordinate directions in \mathcal{S}' , say $\tilde{Q}_\ell = \tilde{Q}(Y_{\ell 0})$, $\ell = 0, 1, \dots, n$, we obtain

$$d\mu_{t_0}|_{(\tilde{Q}_\ell)_{\ell=0}^n} = \prod_{\ell=0}^n \frac{1}{\sqrt{2\pi}} |z_\ell(t_0)|^{-1} \exp\left(-\frac{1}{2}|z_\ell(t_0)|^{-2}\tilde{Q}_\ell^2\right) d\tilde{Q}_\ell. \quad (3.14)$$

in terms of the Lebesgue measures $d\tilde{Q}_\ell$ [25].

Now, we will prove that the support of the measure is smaller than \mathcal{S}' . Concretely, it is given by the topological dual of the subspace of symmetric functions in the Sobolev space $H^\epsilon(\mathbb{S}^2)$ on the 2-sphere, for any $\epsilon > 0$. With this aim, we will use the Bochner-Minlos theorem that plays a key role in the characterization of measures on functional spaces, closely relying on the analysis developed in [26]. We first point out that the space of test functions \mathcal{S} is topologically isomorphic to $\varsigma = \bigcap_{r \in \mathbb{Q}} \varsigma_r$, where

$$\varsigma_r := \left\{ f = (f_\ell)_{\ell=0}^\infty \mid \|f\|_r^2 := \sum_{\ell=0}^\infty (\ell + 1/2)^{2r} f_\ell^2 < +\infty \right\}, \quad (3.15)$$

endowed with the Fréchet topology induced by the norms $(\|\cdot\|_r)_{r \in \mathbb{Q}}$. As a consequence of the Bochner-Minlos theorem (see the theorem 2.3 of [26]), if the covariance $\check{\mathcal{C}}_{t_0}$ is continuous in the norm associated with some ς_r , then the Gaussian measure μ_{t_0} has support on any set of the form

$$\left\{ f \mid \sum_{\ell=0}^\infty (\ell + 1/2)^{-2r-1-2\epsilon} f_\ell^2 < +\infty, \epsilon > 0 \right\} \subset \bigcup_{r \in \mathbb{Q}} \varsigma_r = \varsigma', \quad (3.16)$$

where ς' is the topological dual¹³ of ς . In particular, given the asymptotic behavior of the z_ℓ functions, it is straightforward to check the continuity in the norm corresponding to $r = -1/2$, i.e.,

$$\langle f, \mathcal{C}(t_0)f \rangle \leq N(t_0) \sum_{\ell=0}^\infty (\ell + 1/2)^{-1} f_\ell^2 \quad (3.17)$$

¹² This equation corresponds to the expectation value (3.11) evaluated for $\lambda = (0, f)$ that must coincide with the integral $\int_{\mathcal{S}'} \bar{\Psi}_0^{(t_0)}(\exp(i\hat{Q}_{t_0}[f])\Psi_0^{(t_0)}) d\mu_{t_0}$, where $\Psi_0^{(t_0)} \in \mathcal{H}_s(t_0)$ is the normalized vacuum state.

¹³ Here, $g \in \varsigma'$ is associated with the linear functional $L_g(f) := \sum_{\ell=0}^\infty f_\ell g_\ell$, $f \in \varsigma$.

for certain constant $N(t_0) \in \mathbb{R}^+$. According to this result, the measure μ_{t_0} is concentrated on the set (3.16) for $r = -1/2$, which can be identified with the topological dual \mathfrak{h}'_ϵ of the subspace of symmetric functions in the Sobolev space $H^\epsilon(\mathbb{S}^2)$, for any $\epsilon > 0$,

$$\mathfrak{h}_\epsilon := \left\{ f \in H^\epsilon(\mathbb{S}^2) \mid \mathcal{L}_\sigma f = 0, \|f\|_\epsilon^2 := \sum_{\ell=0}^{\infty} (\ell + 1/2)^{2\epsilon} f_\ell^2 < +\infty \right\}, \quad \epsilon > 0, \quad (3.18)$$

where f_ℓ are the Fourier coefficients of the function f . Therefore, the typical field configurations are not as singular as the delta functions or their derivatives. However, the subset $\mathfrak{b} \subset \mathfrak{h}'_\epsilon$ of symmetric $L^2(\mathbb{S}^2)$ functions has also measure zero. Indeed, consider the characteristic function $\chi_{\mathfrak{b}}$ of the measurable set \mathfrak{b} , defined by

$$\chi_{\mathfrak{b}}[\tilde{Q}] := \lim_{\alpha \rightarrow +0} \exp \left(-\alpha \sum_{\ell=0}^{\infty} \tilde{Q}_\ell^2 \right), \quad (3.19)$$

so that $\chi_{\mathfrak{b}}[\tilde{Q}] = 1$, for $\tilde{Q} \in \mathfrak{b}$, and vanishes anywhere else. Making use of the restriction (3.14), and applying the Lebesgue monotone convergence theorem, it is straightforward to obtain

$$\mu_{t_0}(\mathfrak{b}) = \int_{\mathcal{S}'} \chi_{\mathfrak{b}}[\tilde{Q}] d\mu_{t_0}[\tilde{Q}] = \lim_{\alpha \rightarrow +0} \lim_{n \rightarrow \infty} \prod_{\ell=0}^n \frac{1}{\sqrt{1 + 2\alpha |z_\ell(t_0)|^2}}. \quad (3.20)$$

The limit of the product vanishes as $n \rightarrow \infty$ because of the nonconvergence of the series $\sum_{\ell=0}^{\infty} \log(1 + 2\alpha |z_\ell(t_0)|^2)$, and hence¹⁴ $\mu_{t_0}(\mathfrak{b}) = 0$.

C. Canonical commutation relations

By virtue of the interrelation between operator representation and measures, the representation of the basic momentum observables is [18]

$$\begin{aligned} \pi_s(t_0) \cdot W(\lambda)|_{\lambda=(-g,0)} &= \exp(i\hat{P}_{t_0}[g]), \\ (\hat{P}_{t_0}[g]\Psi)[\tilde{Q}] &= -i(D_{\tilde{Q}}\Psi)[g] - i\tilde{Q}(B^{-1}(t_0)(\mathbf{1} - iA(t_0))g)\Psi[\tilde{Q}], \end{aligned} \quad (3.21)$$

where $\tilde{Q} \in \mathcal{S}'$, $\Psi \in \mathcal{D}_{\hat{P}_{t_0}[g]} \subset \mathcal{H}_s(t_0)$, and $(D_{\tilde{Q}}\Psi)$ denotes the directional derivative of the functional Ψ in the direction defined by $\tilde{Q} \in \mathcal{S}'$, which will acquire a definite sense in terms of the modes \tilde{Q}_ℓ . Note the appearance of the multiplicative term in the momentum operator that depends both on the measure μ_{t_0} —uniquely characterized by the operator $B(t_0)$ —and the operator $A(t_0)$. It guarantees that the momentum operator is symmetric with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{H}_s(t_0)}$. Indeed, just by using the Gaussian integration by parts formula $\int_{\mathcal{S}'} (D_{\tilde{Q}}\Psi)[f] d\mu_{t_0}[\tilde{Q}] = \int_{\mathcal{S}'} \tilde{Q}(\mathcal{C}^{-1}(t_0)f)\Psi[\tilde{Q}] d\mu_{t_0}[\tilde{Q}]$ that can be easily deduced from (3.14),

¹⁴ Since $\mathcal{S} \hookrightarrow \mathfrak{b}$, we have that, as usual for a field theory, the measure μ_{t_0} is not supported on the classical configuration space \mathcal{S} . This is precisely the reason why a suitable distributional extension of \mathcal{S} must be chosen as measure space in order to construct the L^2 space for the Schrödinger representation.

we obtain

$$\begin{aligned}
\langle \Phi | \hat{P}_{t_0}[g] \Psi \rangle_{\mathcal{H}_s(t_0)} &= i \langle (D_{\tilde{Q}} \Phi)[g] | \Psi \rangle_{\mathcal{H}_s(t_0)} + i \langle \Phi | \tilde{Q}(B^{-1}(t_0)(\mathbf{1} + iA(t_0))g) \Psi \rangle_{\mathcal{H}_s(t_0)} \\
&= i \langle (D_{\tilde{Q}} \Phi)[g] + \tilde{Q}(B^{-1}(t_0)(\mathbf{1} - iA(t_0))g) \Phi | \Psi \rangle_{\mathcal{H}_s(t_0)} \\
&= \langle \hat{P}_{t_0}[g] \Phi | \Psi \rangle_{\mathcal{H}_s(t_0)}, \quad \forall \Phi, \Psi \in \mathcal{D}_{\hat{P}_{t_0}[g]}.
\end{aligned}$$

Let us denote $\hat{Q}_\ell(t_0) := \hat{Q}_{t_0}[Y_{\ell 0}]$ and $\hat{P}_\ell(t_0) := \hat{P}_{t_0}[Y_{\ell 0}]$, where the $\hat{Q}_{t_0}[f]$ operator was defined in (3.12). By considering the normalization condition (2.7) and equation (3.10), we get

$$(B^{-1}(t_0)(\mathbf{1} - iA(t_0))Y_{\ell 0})(s) = i \frac{\dot{\bar{z}}_\ell(t_0)}{\bar{z}_\ell(t_0)} Y_{\ell 0}(s),$$

and hence we finally obtain

$$\hat{Q}_\ell(t_0)\Psi = \tilde{Q}_\ell\Psi, \quad \hat{P}_\ell(t_0)\Psi = -i \frac{\partial \Psi}{\partial \tilde{Q}_\ell} + \frac{\dot{\bar{z}}_\ell(t_0)}{\bar{z}_\ell(t_0)} \tilde{Q}_\ell\Psi, \quad (3.22)$$

where Ψ is a functional of the components \tilde{Q}_ℓ . The canonical commutation relations $[\hat{Q}_\ell(t_0), \hat{P}_{\ell'}(t_0)] = i\delta(\ell, \ell')\mathbb{I}$ and $[\hat{Q}_\ell(t_0), \hat{Q}_{\ell'}(t_0)] = 0 = [\hat{P}_\ell(t_0), \hat{P}_{\ell'}(t_0)]$ are obviously satisfied on the appropriate domains.

It is possible to relate the Fock and Schrödinger representations through the action of the annihilation and creation operators on wave functionals [18]. Making use of equations (3.7) and (3.22), we get

$$\hat{a}_\ell(t_0) = \bar{z}_\ell(t_0) \frac{\partial}{\partial \tilde{Q}_\ell}, \quad \hat{a}_\ell^\dagger(t_0) = -z_\ell(t_0) \frac{\partial}{\partial \tilde{Q}_\ell} + \frac{1}{\bar{z}_\ell(t_0)} \tilde{Q}_\ell. \quad (3.23)$$

In particular, the vacuum state is given by the unit constant functional (up to multiplicative phase)

$$\Psi_0^{(t_0)}[\tilde{Q}] = 1, \quad \forall \tilde{Q} \in \mathcal{S}'.$$

There exists then a map $\hat{T}_{t_0} : \mathcal{F}_{\mathcal{P}} \rightarrow \mathcal{H}_s(t_0)$ that unitarily connects the creation and annihilation operators of the Fock and Schrödinger representations [25]. Given the annihilation and creation operators associated with the modes $z_\ell Y_{\ell 0}$, \hat{a}_ℓ and \hat{a}_ℓ^\dagger respectively, the expressions (3.23) correspond to $\hat{T}_{t_0} \circ \hat{a}_\ell \circ \hat{T}_{t_0}^{-1}$ and $\hat{T}_{t_0} \circ \hat{a}_\ell^\dagger \circ \hat{T}_{t_0}^{-1}$, respectively. These relations, and the action $\Psi_0^{(t_0)} = \hat{T}_{t_0}|0\rangle_{\mathcal{P}}$ on the Fock vacuum state $|0\rangle_{\mathcal{P}} \in \mathcal{F}_{\mathcal{P}}$, univocally characterize the unitary transformation \hat{T}_{t_0} .

The general procedure that we have followed guarantees the self-adjointness of the configuration and momentum operators. Indeed, by the successive action of the creation operator on the vacuum state $\Psi_0^{(t_0)}$, we obtain the N -particle states in the Schrödinger representation. These states define, for $N < \infty$, a common, invariant, dense domain of analytic vectors for the configuration and momentum operators, so that their essential self-adjointness is guaranteed, and hence the existence of unique self-adjoint extensions (see Nelson's analytic vector theorem in [27]).

Finally, the probabilistic interpretation of the models is given by the usual Born's corres-

pendence rules [28]. Concretely, given $f \in \mathcal{S}$, the theoretical probability that a measurement carried out in the state Ψ at certain time to determine the value of $\tilde{Q}(f)$ will yield a result contained in the Borel set $\Delta \in \text{Bor}(\mathbb{R})$ for some $\tilde{Q} \in \mathcal{S}'$ is given by

$$P_{\Psi}^{\hat{Q}_{t_0}[f]}(\Delta) = \|\Psi\|_{\mathcal{H}_s(t_0)}^{-2} \langle \Psi | E^{\hat{Q}_{t_0}[f]}(\Delta) \Psi \rangle_{\mathcal{H}_s(t_0)} = \|\Psi\|_{\mathcal{H}_s(t_0)}^{-2} \int_{V_{f,\Delta}} |\Psi[\tilde{Q}]|^2 d\mu_{t_0}[\tilde{Q}], \quad (3.24)$$

where $E^{\hat{Q}_{t_0}[f]}(\Delta)$ is the spectral measure univocally associated with $\hat{Q}_{t_0}[f]$, defined by $(E^{\hat{Q}_{t_0}[f]}(\Delta)\Psi)[\tilde{Q}] = \chi_{V_{f,\Delta}}[\tilde{Q}] \Psi[\tilde{Q}]$, with $\chi_{V_{f,\Delta}}$ being the characteristic function of the measurable set $V_{f,\Delta} := \{\tilde{Q} \in \mathcal{S}' \mid \tilde{Q}(f) \in \Delta\} \in \sigma(\text{Cyl}(\mathcal{S}'))$. $\|\cdot\|_{\mathcal{H}_s(t_0)}$ denotes the norm associated with the inner product $\langle \cdot | \cdot \rangle_{\mathcal{H}_s(t_0)}$. According to this, the measure μ_{t_0} admits the following physical interpretation: it defines the probability measure (3.24) for the vacuum state $\Psi_0^{(t_0)}$.

IV. UNITARY EQUIVALENCE OF SCHRÖDINGER REPRESENTATIONS

Denote by $\tau_{(t_0,t_1)} := \mathfrak{J}_{t_1}^{-1} \circ \mathfrak{J}_{t_0} : \mathbf{P} \rightarrow \mathbf{P}$, $t_1 > t_0$, the symplectomorphism that (i) takes Cauchy data on the embedding $\iota_{t_0}(\mathbb{S}^2) \subset (0, \pi) \times \mathbb{S}^2$; (ii) evolves them to obtain the corresponding solution in \mathcal{S} ; and (iii) finally finds the Cauchy data that this solution induces on the embedding $\iota_{t_1}(\mathbb{S}^2) \subset (0, \pi) \times \mathbb{S}^2$. This map implements the classical time evolution from the embedding $\iota_{t_0}(\mathbb{S}^2)$ to $\iota_{t_1}(\mathbb{S}^2)$ on the canonical phase space, inducing a one-parameter family of states on the Weyl algebra: Let $\alpha_{(t_0,t_1)} : \mathcal{A} \rightarrow \mathcal{A}$ be the $*$ -automorphism associated with the symplectic transformation $\tau_{(t_0,t_1)}$, defined by $\alpha_{(t_0,t_1)} \cdot W(\lambda) := W(\tau_{(t_0,t_1)}(\lambda))$; the dynamical evolution of states in the algebraic formulation of the theory is then given by $\varpi_{t_1} = \varpi_{t_0} \circ \alpha_{(t_0,t_1)}^{-1}$ (Schrödinger picture), with ϖ_{t_0} defined in equation (3.11). The evolved state ϖ_{t_1} acts on the elementary observables as $\varpi_{t_1}(W(\lambda)) = \exp(-\omega(J_{t_1}(\lambda), \lambda)/4)$, where the complex structure

$$J_{t_1} := \tau_{(t_0,t_1)} \circ J_{t_0} \circ \tau_{(t_0,t_1)}^{-1} = \mathfrak{J}_{t_1}^{-1} \circ J_{\mathcal{P}} \circ \mathfrak{J}_{t_1} : \mathbf{P} \rightarrow \mathbf{P}$$

defines a new Schrödinger representation¹⁵ $\mathcal{H}_s(t_1)$. Clearly, the condition of unitary equivalence of the Schrödinger representations corresponding to different values $t_0 < t_1$ of the time parameter amounts to demanding the unitary implementability of the symplectic transformation $\tau_{(t_0,t_1)}$ in the $\mathcal{H}_s(t_0)$ representation¹⁶. In that case, there exists a unitary transformation $\hat{V}_{(t_0,t_1)} : \mathcal{H}_s(t_0) \rightarrow \mathcal{H}_s(t_1)$ mapping the configuration and momentum operators from one representation into the other, in such a way that

$$\begin{aligned} \hat{V}_{(t_0,t_1)} \circ \hat{a}_{\ell}(t_0) \circ \hat{V}_{(t_0,t_1)}^{-1} &= \alpha_{\ell}(t_0, t_1) \hat{a}_{\ell}(t_1) + \beta_{\ell}(t_0, t_1) \hat{a}_{\ell}^{\dagger}(t_1), \\ \hat{V}_{(t_0,t_1)} \circ \hat{a}_{\ell}^{\dagger}(t_0) \circ \hat{V}_{(t_0,t_1)}^{-1} &= \bar{\beta}_{\ell}(t_0, t_1) \hat{a}_{\ell}(t_1) + \bar{\alpha}_{\ell}(t_0, t_1) \hat{a}_{\ell}^{\dagger}(t_1), \end{aligned} \quad (4.1)$$

¹⁵ Here, we will make a notational abuse and simply denote the triplet $(\mathcal{H}_s(t), \pi_s(t), \Psi_0^{(t)})$ as $\mathcal{H}_s(t)$.

¹⁶ In this way, $J_{t_1} - J_{t_0}$ is a Hilbert-Schmidt operator in the one-particle Hilbert space constructed from J_{t_0} (or equivalently J_{t_1}).

where

$$\alpha_\ell(t_0, t_1) := i\left(\bar{z}_\ell(t_0)\dot{z}_\ell(t_1) - z_\ell(t_1)\dot{\bar{z}}_\ell(t_0)\right), \quad \beta_\ell(t_0, t_1) := i\left(\bar{z}_\ell(t_0)\dot{\bar{z}}_\ell(t_1) - \bar{z}_\ell(t_1)\dot{z}_\ell(t_0)\right). \quad (4.2)$$

According to the results achieved in [16], once we consider an $SO(3)$ invariant complex structure $J_{\mathcal{P}}$ verifying the conditions (2.12), the quantum dynamics can be unitarily implemented in $\mathcal{F}_{\mathcal{P}}$, i.e., there exists a unitary operator $\hat{U}(t, t') : \mathcal{F}_{\mathcal{P}} \rightarrow \mathcal{F}_{\mathcal{P}}$ encoding the information about the evolution of the system from time t to t' . This condition is precisely ensured by the square summability of the β_ℓ coefficients appearing in the Bogoliubov transformation (4.1), and guarantees that the map $\hat{V}_{(t_0, t_1)}$ is well defined, i.e., the Schrödinger representations corresponding to different times t_0, t_1 are equivalent. The unitary transformation $\hat{V}_{(t_0, t_1)} = \hat{T}_{t_1} \circ \hat{U}(t_0, t_1) \circ \hat{T}_{t_0}^{-1}$ relating them is completely characterized by the relations (4.1) and the action on the vacuum state $\Psi_0^{(t_0)} \in \mathcal{H}_s(t_0)$, given by

$$\left(\hat{V}_{(t_0, t_1)}\Psi_0^{(t_0)}\right)[\tilde{Q}] = \prod_{\ell=0}^{\infty} \frac{|z_\ell(t_1)|^{1/2}}{|z_\ell(t_0)|^{1/2}} \exp\left(-\frac{1}{2} \frac{\beta_\ell(t_0, t_1)}{\bar{z}_\ell(t_0)\bar{z}_\ell(t_1)} \tilde{Q}_\ell^2\right) \in \mathcal{H}_s(t_1), \quad (4.3)$$

where we have used the fact that $\hat{a}_\ell(t_0)\Psi_0^{(t_0)} = 0$, $\forall \ell \in \mathbb{N} \cup \{0\}$, and the expressions (2.7), (3.23) and (4.1) to obtain the differential equations verified by this state; namely, $\partial \hat{V}_{(t_0, t_1)}\Psi_0^{(t_0)} / \partial \tilde{Q}_\ell = -(\beta_\ell(t_0, t_1) / \bar{z}_\ell(t_0)\bar{z}_\ell(t_1)) \tilde{Q}_\ell \hat{V}_{(t_0, t_1)}\Psi_0^{(t_0)}$, $\ell \in \mathbb{N} \cup \{0\}$. The equation (4.3) must be interpreted as the limit in the $\mathcal{H}_s(t_1)$ -norm of the Cauchy sequence of normalized vectors $f_n \in \mathcal{H}_s(t_1)$ obtained by extending the product (4.3) to a finite integer $n \in \mathbb{N}$.

The mutual absolute continuity of any two Gaussian measures associated with different times $t_0, t_1 \in (0, \pi)$ is also verified¹⁷, i.e. they have the same zero measure sets. This property requires that the operator $\mathcal{C}(t_1) - \mathcal{C}(t_0)$ is Hilbert-Schmidt [29, 30, 31], which is satisfied in our case. Indeed, it is straightforward to check that the sequence $(|z_\ell(t_1)|^2 - |z_\ell(t_0)|^2)_{\ell=0}^{\infty}$ is square summable. On the contrary, for the original scalar field $\phi = \xi/\sqrt{\sin t}$, for which the time evolution is not unitary, we get the nonequivalence of the representations obtained for different times, and also the impossibility of such continuity. In this case, the mutual singularity of measures can be expected, as was proved for the vacuum Gowdy \mathbb{T}^3 model in [13].

Note that the map $\hat{T}_{t_0} : \mathcal{F}_{\mathcal{P}} \rightarrow \mathcal{H}_s(t_0)$ introduced in subsection III C does not connect the configuration and momentum operators of the Fock representation, $\hat{Q}_\ell(t) = z_\ell(t)\hat{a}_\ell + \bar{z}_\ell(t)\hat{a}_\ell^\dagger$ and $\hat{P}_\ell(t) = \dot{z}_\ell(t)\hat{a}_\ell + \dot{\bar{z}}_\ell(t)\hat{a}_\ell^\dagger$, respectively, with those of the Schrödinger one (except for $t = t_0$). However, owing to the unitary implementability of the dynamics, there exists also a unitary transformation $\hat{V}_{\mathcal{F}_{\mathcal{P}}, t_0}(t) : \mathcal{F}_{\mathcal{P}} \rightarrow \mathcal{H}_s(t_0)$, such that $\hat{V}_{\mathcal{F}_{\mathcal{P}}, t_0}(t) \circ \hat{a}_\ell \circ \hat{V}_{\mathcal{F}_{\mathcal{P}}, t_0}^{-1}(t) = \alpha_\ell(t, t_0)\hat{a}_\ell(t_0) + \beta_\ell(t, t_0)\hat{a}_\ell^\dagger(t_0)$, $\hat{V}_{\mathcal{F}_{\mathcal{P}}, t_0}(t) \circ \hat{a}_\ell^\dagger \circ \hat{V}_{\mathcal{F}_{\mathcal{P}}, t_0}^{-1}(t) = \bar{\beta}_\ell(t, t_0)\hat{a}_\ell(t_0) + \bar{\alpha}_\ell(t, t_0)\hat{a}_\ell^\dagger(t_0)$, relating these operators. In terms of the unitary evolution operator on $\mathcal{F}_{\mathcal{P}}$, we have $\hat{V}_{\mathcal{F}_{\mathcal{P}}, t_0}(t) = \hat{T}_{t_0} \circ \hat{U}^{-1}(t_0, t)$. Finally, given the quantum Hamiltonian (2.22) in the Fock representation, with dense domain $\mathcal{D}_{\hat{H}_0(t)} \subset \mathcal{F}_{\mathcal{P}}$ spanned by the states with a finite number of particles, the

¹⁷ It is possible to show that the equivalence of measures is a necessary condition for the unitary equivalence between Schrödinger representations [14].

corresponding operator in the $\mathcal{H}_s(t_0)$ representation is given by $\hat{V}_{\mathcal{F}_{\mathcal{P}},t_0}(t) \circ \hat{H}_0(t) \circ \hat{V}_{\mathcal{F}_{\mathcal{P}},t_0}^{-1}(t)$,

$$\frac{1}{2} \sum_{\ell=0}^{\infty} \left[-\frac{\partial^2}{\partial \tilde{Q}_\ell^2} - 2i \frac{\dot{\tilde{z}}_\ell(t_0)}{\tilde{z}_\ell(t_0)} \tilde{Q}_\ell \frac{\partial}{\partial \tilde{Q}_\ell} + \left(\frac{\dot{\tilde{z}}_\ell^2(t_0)}{\tilde{z}_\ell^2(t_0)} + \frac{1}{4}(1 + \csc^2 t) + \ell(\ell+1) \right) (\tilde{Q}_\ell^2 - |z_\ell(t_0)|^2) \right]$$

modulo an irrelevant real term proportional to the identity. Note, by contrast, that the complex independent term appearing in the previous expression is necessary to ensure that the operator is self-adjoint. This Hamiltonian is defined in the dense subspace $\hat{V}_{\mathcal{F}_{\mathcal{P}},t_0}(t) \mathcal{D}_{\hat{H}_0(t)} = \{ \hat{V}_{\mathcal{F}_{\mathcal{P}},t_0}(t) f \mid f \in \mathcal{D}_{\hat{H}_0(t)} \} \subset \mathcal{H}_s(t_0)$ generated by the cyclic vector $\hat{V}_{\mathcal{F}_{\mathcal{P}},t_0}(t) |0\rangle_{\mathcal{P}} \in \mathcal{H}_s(t_0)$.

V. COMMENTS

We have constructed the Schrödinger representation for the linearly polarized Gowdy $\mathbb{S}^1 \times \mathbb{S}^2$ and \mathbb{S}^3 models coupled to massless scalar fields in a mathematically rigorous and self-contained way, completing in this way the quantization of these systems given in [16]. We have assumed the use of the redefined fields for which the dynamics is well defined and unitary. As proved in [16, 17], the complex structures $J_{\mathcal{P}}$ verifying the conditions (2.12) lead to unitarily equivalent quantum theories, and hence the Schrödinger representations corresponding to them are also equivalent. Note that, as far as the support of the measure or the unitary implementability of the dynamics is concerned, the discussions and results obtained for these models are analogous to those found for the vacuum \mathbb{T}^3 model in [13] and [14]. It could be argued that this similarity is somehow expected due to the fact that the critical features of the systems are determined by their ultraviolet behaviors, and these should not be sensitive to the topology of the spacetimes. This argument can be found, for example, in [32] concerning the simplest generalization of Minkowski space quantum field theory to the $\mathbb{R} \times \mathbb{T}^3$ spacetime with closed spatial sections. This compactification can modify the long-wavelength behavior of the system, but not the ultraviolet one, so that both spacetimes suffer from the same ultraviolet divergence properties. Such statement is clearly intuitive, but it is not obvious to what extent it is true for quantum field theories in spacetimes, like those corresponding to the Gowdy models, that are not locally isometric. In this respect, the similarity of the results is probably due to the similar structure of the differential equations verified by the mode functions. In any case, it is interesting to analyze in a rigorous way the particularities of the quantizations for the different topologies.

Finally, it is important to highlight the advantage of using the re-scaled fields that make the quantum dynamics unitary, given that in this case it is possible to obtain a unique (up to unitary equivalence) Fock/Schrödinger representation for these models. As a direct consequence, the mutual absolute continuity of the measures corresponding to different times is verified. Neither of these properties can be attained for the original variables. In this situation, even if the failure of the unitarity of time evolution and the mutual singularity of measures are not serious obstacles for a suitable probabilistic interpretation of the models [7, 13], we must face the lack of uniqueness of the representation.

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- [1] R. H. Gowdy, Phys. Rev. Lett. **27**, 826 (1971); Ann. Phys. **83**, 203 (1974).
 - [2] C. W. Misner, Phys. Rev. D **8**, 3271 (1973).
 - [3] B. K. Berger, Ann. Phys. **83**, 458 (1974); Phys. Rev. D **11**, 2770 (1975).
 - [4] G. A. Mena Marugán, Phys. Rev. D **56**, 908 (1997).
 - [5] M. Pierri, Int. J. Mod. Phys. D **11**, 135 (2002).
 - [6] A. Corichi, J. Cortez, and H. Quevedo, Int. J. Mod. Phys. D **11**, 1451 (2002).
 - [7] C. G. Torre, Phys. Rev. D **66**, 084017 (2002).
 - [8] J. Cortez and G. A. Mena Marugán, Phys. Rev. D **72**, 064020 (2005).
 - [9] A. Corichi, J. Cortez, and G. A. Mena Marugán, Phys. Rev. D **73**, 084020 (2006).
 - [10] J. F. Barbero G., D. G. Vergel, and E. J. S. Villaseñor, Phys. Rev. D **74**, 024003 (2006).
 - [11] A. Corichi, J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Class. Quant. Grav. **23**, 6301 (2006).
 - [12] J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Phys. Rev. D **75**, 084027 (2007).
 - [13] C. G. Torre, Class. Quant. Grav. **24**, 1 (2007).
 - [14] A. Corichi, J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Phys. Rev. D **76**, 124031 (2007).
 - [15] J. F. Barbero G., D. G. Vergel, and E. J. S. Villaseñor, Class. Quant. Grav. **24**, 5945 (2007).
 - [16] J. F. Barbero G., D. G. Vergel, and E. J. S. Villaseñor, Class. Quant. Grav. **25**, 085002 (2008).
 - [17] J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Class. Quant. Grav. **25**, 105005 (2008).
 - [18] A. Corichi, J. Cortez, and H. Quevedo, Phys. Rev. **D66**, 085025 (2002); Class. Quant. Grav. **20**, L83 (2003); Annals Phys. **313**, 446-478 (2004).
 - [19] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. The University of Chicago Press, USA (1994).
 - [20] J. F. Barbero G., D. G. Vergel, and E. J. S. Villaseñor, in preparation.
 - [21] D. Shale, Trans. Am. Math. Soc. **103**, 149 (1962).
 - [22] D. A. Dubin and M. A. Hennings, *Quantum Mechanics, Algebras and Distributions*. Longman Scientific & Technical (1990).
 - [23] A. Ashtekar, Commun. Math. Phys. **71**, 59-64 (1980).
 - [24] A. Ashtekar and A. Magnon, Proc. Roy. Soc. Lond. **A346**, 375 (1975).
 - [25] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View* (Second Edition). Springer (1984).
 - [26] B. Simon, *Functional Integration and Quantum Physics*. Academic Press, New York, (1974).
 - [27] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I*. Academic Press, New York, (1980).

- [28] E. Prugovečki, *Quantum Mechanics in Hilbert Space* (Second Edition). Academic Press, New York, (1981).
- [29] B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*. Princeton University Press, Princeton, NJ, (1974).
- [30] Y. Yamasaki, *Measures on Infinite Dimensional Spaces*. World Scientific, Singapore, (1985).
- [31] J. M. Mourão, T. Thiemann, and J. M. Velhinho, J. Math. Phys. **40**, 2337 (1999).
- [32] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge University Press, (1982).